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CALCULATION OF THE SECOND VARIATION IN THE PROBLEM OF THE STABILITY OF THE STEADY MOTION OF A RIGID BODY CONTAINING A LIQUID†

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The stability of equilibrium or of steady motion of a rigid body containing a liquid is studied. A theorem due to Rumyantsev is used to derive the sufficient condition for stability corresponding to a minimum of the variable potential energy for the transformed rigid body. A procedure is presented for expressing the second variation of the variable potential energy as a quadratic form in the parameters that define the position of the body. The calculations are substantially simplified for the problem of the stability of equilibrium or uniform rotation around a fixed axis of a rigid body with liquid in a uniform gravitational field. Rumyantsev's results are derived anew. © 1997 Elsevier Science Ltd. All rights reserved.

1. AUXILIARY FORMULA

Let Ω be a domain boundary bounded by a surface $\partial\Omega$. This domain is transformed to a nearby position Ω', defined by a small displacements $\mathbf{u}$ of the points of $\partial\Omega$.

Let $\Omega' - \Omega$ denote the domain "swept out" by the surface $\partial\Omega$, that is, the set of points $\mathbf{M}^\lambda$ defined by the relationship $\mathbf{OM}^\lambda = \mathbf{OM} + \lambda\mu$, where $\mathbf{M}$ is a point of $\partial\Omega$, $\mathbf{u} = \mathbf{MM}'$ is the displacement of $\mathbf{M}$, λ is a parameter, $0 < \lambda < 1$, and $\mathbf{M}$ and $\mathbf{u}$ depend, say, on curvilinear coordinates α and β defined on $\partial\Omega$ and M^λ depends on α, β and λ.

The difference between $\Omega' - \Omega$ and $(\Omega \cup \Omega') - (\Omega \cup \Omega')$ consists of domains which are at least three orders of magnitude smaller than $|\mathbf{u}|$.

To carry out the calculations, we refer $\partial\Omega$ to curvilinear coordinates α and β defined so that the vector product $\mathbf{M}_\alpha \times \mathbf{M}_\beta$ points along the normal to Ω. Throughout this paper, the subscripts α and β denote the appropriate partial derivatives.

Let w be a sufficiently regular function whose range of definition contains both Ω and Ω'.

By our previous remark, if we confirm ourselves to terms of order less than or equal to two relative to $|\mathbf{u}|$, we can write

$$\int_{\Omega'-\Omega} w d\tau = \int_{\Omega'-\Omega} w(\mathbf{M}+\lambda\mathbf{u})(\mathbf{M}_\alpha^\lambda, \mathbf{M}_\beta^\lambda, \mathbf{M}_\lambda^\lambda) d\alpha d\beta d\lambda = \int_{\Omega'-\Omega} w(\mathbf{M})(\mathbf{M}_\alpha \times \mathbf{M}_\beta)\cdot \mathbf{u} d\alpha d\beta d\lambda +$$

$$+ \int_{\Omega'-\Omega} \lambda\{(\operatorname{grad} w \cdot \mathbf{u})(\mathbf{M}_\alpha, \mathbf{M}_\beta, \mathbf{u}) + w(\mathbf{M})(\mathbf{M}_\alpha \times \mathbf{M}_\beta + \mathbf{u}_\alpha \times \mathbf{M}_\beta)\cdot \mathbf{u}\} d\alpha d\beta d\lambda$$

(the subscript λ denotes partial differentiation with respect to λ).

Observing that the domain $\Omega' - \Omega$ may be identified with $\partial\Omega \times]0, 1[$ and that the area element dS on the surface $\partial\Omega$ is $AB d\alpha d\beta$, we see that, up to terms of higher than second order in $|\mathbf{u}|$

$$\int_{\Omega'-\Omega} w d\tau = \int_{\partial\Omega} w(\mathbf{M}) u_n dS + \frac{1}{2}\int_{\partial\Omega} [(\operatorname{grad} w \cdot \mathbf{u}) u_n + w(\mathbf{u}\cdot\mathbf{n}_1)] dS \tag{1.1}$$

where we have introduced the notation

$$u_n = \mathbf{u}\cdot\mathbf{n}, \quad \mathbf{n} = \frac{\mathbf{M}_\alpha \times \mathbf{M}_\beta}{AB}, \quad \mathbf{n}_1 = \frac{1}{AB}(\mathbf{M}_\alpha \times \mathbf{M}_\beta + \mathbf{u}_\alpha \times \mathbf{M}_\beta)$$
$$A = |\mathbf{M}_\alpha|, \quad B = |\mathbf{M}_\beta|$$

In particular, setting $w = 1$, we obtain the first and second variation of the volume Ω

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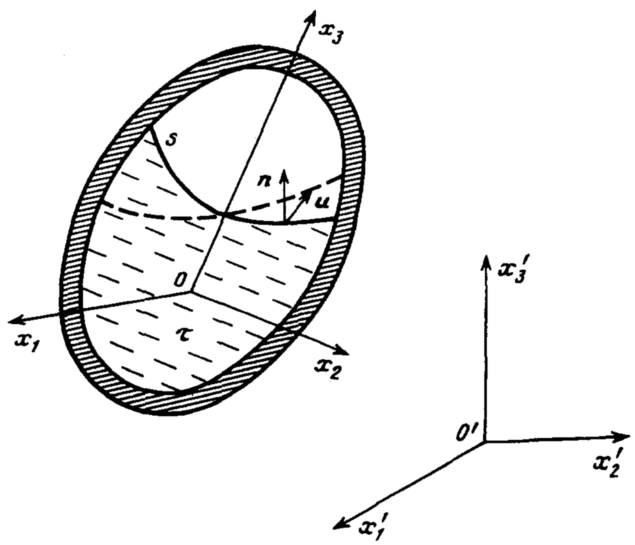

Fig. 1.

$$\int_{\partial\Omega} u_n dS + \frac{1}{2}\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})dS$$

Throughout what follows, we retain the notation of [1], Chap. IV.

2. CALCULATION OF THE FIRST AND SECOND VARIATION OF THE VARIED POTENTIAL ENERGY FOR THE STEADY MOTION OF A RIGID BODY CONTAINING A LIQUID

Let us consider an absolutely rigid body with a simply-connected cavity of arbitrary shape containing an incompressible homogeneous ideal liquid (see Fig. 1). The position of the body and the liquid relative to a fixed system of coordinates $O'x'_1x'_2x'_3$ will be defined by the coordinates of the body q_j ($j = 1, \ldots, n-1$) and the absolute coordinates x'_1, x'_2, x'_3 or relative coordinates x_1, x_2, x_3 of the liquid particles. Suppose that stationary constraints imposed on the system allow the body to rotate about the x'_3 axis, while the given forces acting on the liquid particles admit of force functions $U_1(q_j)$ and $U_2(x_1, x'_2, x'_3)$ and do not produce a torque about other x'_3 axis. Then an energy integral and an area integral exist for the plane orthogonal to the x'_3 axis, and the variable potential energy is

$$W = \frac{k_0^2}{2I} - U_1 - \rho\int U_2 d\tau$$

where k_0 is the value of the area constant k for uniform rotation of the entire system as a single rigid body about the x'_3 axis at angular velocity ω, I is the moment of inertia of the system about the x'_3 axis, and ρ is the density of the liquid; throughout, the volume integrals are evaluated over the domain τ occupied by the liquid.

The equations of steady motion are obtained from the requirement that $\delta W = 0$ provided that the volume of the liquid is constant up to first-order terms. Calculating δW, we obtain the well-known equations

$$\frac{k_0^2}{2I_0}\frac{\partial I}{\partial q_j} + \frac{\partial U_1}{\partial q_j} + \rho\int \frac{\partial U_2}{\partial q_j} d\tau = 0 \quad (j = 1, \ldots, n-1) \tag{2.1}$$

for the coordinates q_j of the rigid body in steady motion, and the equation

$$\frac{1}{2}\omega^2(x_1'^2 + x_2'^2) + U_2(x_1', x_2', x_3') = c_0 \left(\omega = \frac{k_0}{I_0}\right)$$

of the free surface S of the liquid in steady motion. Here I_0 is the value of I for steady motion and the constant c_0 is defined by the quantity of liquid in the cavity.

Let us calculate the second variation W on changing from the configuration corresponding to steady motion, for which all the q_j vanish, to a nearby configuration. We impart the displacement to the system as a single rigid body; the free surface of the liquid occupies a position S, and we then displace the liquid to a new position (denoting the displacement of a point of S by $\mathbf{u}$). In this problem the domain "swept out" by the surface $\partial\tau$ bounding τ coincides up to infinitesimals of order at least three with the domain swept out by S; we must therefore replace $\partial\Omega$ in (1.1) (here $\partial\tau$) by S. Throughout what follows, surface integrals are evaluated over the surface S. Thus, we have

$$\delta^2 W = \frac{k_0^2}{2}\left[\frac{(\delta I)^2}{I_0^3} - \frac{\delta^2 I}{I_0^2}\right] - \frac{1}{2}\sum_{i,j} \frac{\partial^2 U_1}{\partial q_i \partial q_j} q_i q_j - \frac{1}{2}\rho\int \sum_{i,j} \frac{\partial^2 U_2}{\partial q_i \partial q_j} q_i q_j d\tau -$$
$$-\frac{1}{2}\rho\left[\iint [\operatorname{grad} U_2 \cdot \mathbf{u})u_n + U_2(\mathbf{u}\cdot\mathbf{n}_1)\right] dS$$

Let us evaluate δI and $\delta^2 I$. By the previous remarks, we can write

$$I(q_j, \tau) - I(0, \tau_0) = [I(q_j, \tau_0) - I(0, \tau_0)] + [I(q_j, \tau) - I(q_j, \tau_0)]$$

In the new position of the liquid as a rigid body we have, up to fourth-order terms

$$x'^2 = x^2 + \sum_j \frac{\partial x'^2}{\partial q_j} q_j \quad (x'^2 = x_1'^2 + x_2'^2, \quad x^2 = x_1^2 + x_2^2)$$

Consequently, we can write

$$\delta I = \sum_j \frac{\partial I}{\partial q_j} q_j + \rho J, \quad J = \iint x^2 u_n dS$$

$$\delta^2 I = \frac{1}{2}\sum_{i,j} \frac{\partial^2 I}{\partial q_i \partial q_j} q_i q_j + \rho\iint \sum_j \frac{\partial x'^2}{\partial q_j} q_j u_n dS + \frac{1}{2}\rho\iint [(\operatorname{grad} x^2 \cdot \mathbf{u})u_n + x^2(\mathbf{u}\cdot\mathbf{n}_1)] dS$$

A necessary condition for W to have a minimum is $\delta^2 W \geqslant 0$ for all $\mathbf{u}$ such that the volume of the liquid is constant

$$\int\int u_n dS = 0 \text{ in the first approximation,} \tag{2.2}$$

$$\int\int (\mathbf{u}\cdot\mathbf{n}_1) dS = 0 \text{ in the second approximation} \tag{2.3}$$

Taking condition (2.3) into account, as well as the fact that $f_0 = c_0$ on S and $\operatorname{grad} f_0 = -|\operatorname{grad} f_0|\mathbf{n}$, since in steady motion the liquid must be on the side of the free surface where $f_0 > c_0$, we obtain

$$\delta^2 W = -\rho\sum_j \iint \frac{\partial f'}{\partial q_j} q_j u_n dS + \frac{k_0^2}{2I_0^3}\left(\sum \frac{\partial I}{\partial q_j} q_j + \rho J\right)^2 -$$
$$-\frac{1}{2}\sum_{i,j}\left(\frac{k_0^2}{2I_0^3}\frac{\partial^2 I}{\partial q_i \partial q_j} + \frac{\partial^2 U_1}{\partial q_i \partial q_j} + \rho\int \frac{\partial^2 U_2}{\partial q_i \partial q_j} d\tau\right) q_i q_j + \frac{1}{2}\rho\iint |\operatorname{grad} f_0| u_n^2 dS \tag{2.4}$$
$$f_0 = \frac{k_0^2}{2I_0^2} x^2 + U_2(x_i, 0), \quad f' = \frac{k_0^2}{2I_0^2} x'^2 + U_2(x_i, q_j)$$

Note that $\delta^2 W$ depends on the normal component u_n of $\mathbf{u}$.

3. CALCULATION OF THE SECOND VARIATION OF THE VARIED ENERGY POTENTIAL FOR THE TRANSFORMED RIGID BODY

By a theorem due to Rumyantsev [1, Chap. IV, Sec. 4, Theorem VIII], a sufficient condition for the steady motion to be stable may be obtained by determining the minimum of W for the configuration bounded by the surface S'

$$\frac{k_0^2}{2I(q_j, \tau)} x'^2 + U_2(x_i, q_j) = c$$

where the constant c is defined by the amount of liquid in the cavity of the body corresponding to the transformed rigid body.

We will show that if the point x_i describes S, then the point $x_i + u_n n_i$ will describe S' in the first approximation. Indeed, in the first approximation

$$U_2(x_i+u_n n_i,\ q_j)=U_2(x_i,\ 0)+u_n \operatorname{grad} U_2(x_i,\ 0)\cdot \mathbf{n}+\sum_j \left(\frac{\partial U_2(x_i,\ q_j)}{q_j}\right)_0 q_j$$

$$x'^2(x_i+u_n n_i,\ q_j)=x^2+u_n \operatorname{grad} x^2\cdot \mathbf{n}+\sum_j \left(\frac{\partial x'^2}{\partial q_j}\right)_0 q_j$$

On the other hand, we can write

$$\frac{k_0^2}{2I^2(q_j,\ \tau)}=\frac{k_0^2}{2I_0^2}-\frac{k_0^2}{I_0^3}\delta I+\ldots$$

Substituting this into the equation for S', we obtain

$$c-c_0=-|\operatorname{grad} f_0|u_n+\sum_j \left(\frac{\partial f'}{\partial q_j}\right)_0 q_j-\frac{k_0^2}{I_0^3}x^2\left[\sum_j \left(\frac{\partial I}{\partial q_j}\right)_0 q_j+\rho J\right]$$

This equation defines an expression for the component u_n, which depends linearly on $c-c_0$ and the integral J. Multiplying by x^2 and integrating with respect to S, we obtain this integral as a function of q_j and $c-c_0$

$$J=\frac{I_0^3}{K(x_1,\ x_2)}\left[-(c-c_0)\iint QdS+\sum_j \left\{\iint \left[\left(\frac{\partial f'}{\partial q_j}\right)_0-\frac{k_0^2}{I_0^3}x^2\left(\frac{\partial I}{\partial q_j}\right)_0\right]QdS\right\}q_j\right]$$

$$K(x_1,\ x_2)=I_0^3+k_0^2\rho\iint \frac{x^4}{|\operatorname{grad} f_0|}dS,\quad Q(x_1,\ x_2)=\frac{x^2}{|\operatorname{grad} f_0|}$$

Substituting this relationship into the expression for u_n, we obtain

$$u_n=A(x_1,\ x_2)(c-c_0)+\sum_j B_j(x_1,\ x_2)q_j$$

where

$$A(x_1,\ x_2)=\frac{k_0^2 Q}{K(x_1,\ x_2)}\iint QdS-|\operatorname{grad} f_0|^{-1}$$

$$B_j(x_1,\ x_2)=\left(\frac{\partial f''}{\partial q_j}\right)_0|\operatorname{grad} f_0|^{-1}-\frac{k_0^2 Q}{K(x_1,\ x_2)}\iint \left(\frac{\partial f''}{\partial q_j}\right)_0 QdS$$

$$f''=f'-\frac{k_0^2}{I_0^3}x^2 I$$

and x_3 is expressed on S as a function of x_1 and x_2.

Using (2.2), we find that

$$c-c_0=-\sum_j \frac{\iint B_j dS}{\iint AdS}q_j$$

and, finally

$$u_n=\sum_j \left[B_j(x_1,\ x_2)-A(x_1,\ x_2)\frac{\iint B_j dS}{\iint AdS}\right]q_j$$

Thus, u_n is a linear form in q_j, whose coefficients are functions of the coordinates x_1 and x_2 of the points of S.

Substituting the value of u_n into expression (2.4) for $\delta^2 W$, we obtain a quadratic form in the parameters q_j. The requirement that this form be positive definite yields sufficient conditions for the steady motion to be stable.

4. EXAMPLE. STABILITY OF THE UNIFORM ROTATION OF A RIGID BODY WITH A FIXED POINT AND A CAVITY CONTAINING A LIQUID IN A GRAVITATIONAL FIELD ([1], CHAP. IV, SEC. 6)

Consider a rigid body with one fixed point O and a cavity containing a liquid in a uniform gravitational field. We will assume that the x'_3 axis passes through the body's centre of gravity.

Using the notation of [1], we obtain

$$U=-Mg(x_{c1}\gamma_1+x_{c2}\gamma_2+x_{c3}\gamma_3)$$
$$I=A\gamma_1^2+B\gamma_2^2+C\gamma_3^2-2D\gamma_2\gamma_3-2E\gamma_3\gamma_1-2F\gamma_1\gamma_2$$
$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1$$

where M is the mass of the system, x_{ci} (= 1, 2, 3) are the coordinates of its centre of mass, γ is the unit vector along the upward vertical, and A, B, C, D, E and F are the moments of inertia of the system about the x_i axes and the centrifugal moments of inertia.

The equations of steady motion are

$$\frac{1}{2}\omega^2\frac{\partial I}{\partial\gamma_i}+\frac{\partial U}{\partial\gamma_i}=0,\quad i=1,\ 2$$

Let us consider the solution

$$\gamma_1=\gamma_2=0,\quad \gamma_3=1;\quad x_{c1}=x_{c2}=0,\quad x_{c3}=x_{c3}^0;\quad D=E=0$$

on the assumption that $F=0$, so that x_1, x_2 and x_3 are the principal axes of inertia of the system in steady motion.

In this case the free surface S of the liquid is a paraboloid of revolution

$$\tfrac{1}{2}\omega^2x^2-gx_3=c_0$$

where c_0 is a constant which depends on the volume of the liquid.

Let us assume that the projection of the free surface S onto the x_1Ox_2 plane is a circular annulus of radii R_1 and R_2 ($R_1>R_2$). Calculation of δ^2W by formula (2.4) produces an expression identical with that obtained previously [1, p. 208, formula (4.71)].

We now calculate δ^2W for the transformed rigid body. First

$$u_n=-\frac{1}{\omega^2G}\left[c-c_0+(\omega^2x_3+g)(x_1\gamma_1+x_2\gamma_2)+\frac{\rho\omega^2x^2}{I_0}\iint x^2u_n dS\right] \tag{4.1}$$

$$(G=(x^2+\omega^{-4}g^2)^{1/2})$$

Multiplying both sides of this equality by x^2, integrating with respect to S and noting that, by the symmetry of the rotation

$$\iint(\omega^2x_3+g)(x_1\gamma_1+x_2\gamma_2)G^{-1}dS=0$$

we obtain an expression for the integral occurring in (4.1), substitution of which into (4.1) gives

$$u_n=A(x_1,\ x_2)(c-c_0)-(\omega^2x_3+g)(x_1\gamma_1+x_2\gamma_2)/(\omega^2G)$$

Taking condition (2.2) into consideration and again using the symmetry of the rotation, we obtain $c-c_0=0$ and

$$u_n=-\left(\frac{\omega^2}{2g}x^2-\frac{c_0}{g}+\frac{g}{\omega^2}\right)(x_1\gamma_1+x_2\gamma_2)G^{-1}$$

We now return to the expression for δ^2W.

Together with the expression obtained for u_n, we have $\iint x^2u_n dS=0$, and after computing the integral (by changing to polar coordinates in the x_1Ox_2) plane, we obtain Rumyantsev's original formula [1]

$$\delta^2W=-\frac{1}{2}\{[\omega^2(A-C)+Mgx_{c3}^0]\gamma_1^2+[\omega^2(B-C)+Mgx_{c3}^0]\gamma_2^2\}+$$
$$+\frac{1}{2}\pi\rho g\int_{R_2}^{R_1}\left[\frac{\omega^2}{g^2}\left(\frac{1}{2}\omega^2r^2-c_0\right)+1\right]^2r^3dr(\gamma_1^2+\gamma_2^2) \tag{4.2}$$

which enables us to analyse the stability of the steady motion.

Following Rumyantsev, let us consider a case in which the angular velocity of rotation is very large. Then the free surface S of the liquid in steady motion is a circular cylinder $x^2 = b^2$, and after calculations using formula (2.4) we obtain an expression differing from (4.2) in the integral term, which is now

$$\rho\omega^2 \iint x_3(x_1\gamma_1 + x_2\gamma_2)u_n dS + \frac{1}{2}\rho\omega^2 b \iint u_n^2 dS$$

We now calculate $\delta^2 W$ for the transformed rigid body.

The calculation of u_n is simplified. Defining on S

$$x_1 = b\cos\theta, \quad x_2 = b\sin\theta$$

and assuming that S cuts the surface of the cavity in circles with centres on the x_3 axis at points with coordinates $x_3 = h \pm d$, we see, proceeding as before, that

$$u_n = -x_3(\gamma_1\cos\theta + \gamma_2\sin\theta)$$

Substitution into the expression $\delta^2 W$ again yields Rumyantsev's result [1].

5. THE EQUILIBRIUM CASE. EXAMPLE

In the equilibrium case, the computations are much simpler.

The equations of equilibrium are

$$\frac{\partial U_1}{\partial q_j} + \rho\int \frac{\partial U_2}{\partial q_j} d\tau = 0, \quad j = 1, 2, \ldots, n; \quad U_2 = c_0$$

We must set $k_0 = 0$ in expression (2.4) for $\delta^2 W$ and in the formulae of Section 3; then $f_0 = U_2(x_i, 0), f' = f'' = U_2(x_i, q_j)$.

Let us consider Rumyantsev's example of the equilibrium of a rigid body with one fixed point and a cavity containing a liquid in a gravitational field.

Consider the solution

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1; \quad x_{c1} = x_{c2} = 0, \quad x_3 = x_3^0$$

Using the expressions

$$U_2 = -gx_3' = -g(x_1\gamma_1 + x_2\gamma_2 + x_3(1-\gamma_1^2-\gamma_2^2)^{1/2}), \quad U_1 = -M_1 g x_{13}(1-\gamma_1^2-\gamma_2^2)^{1/2}$$

where M_1 is the mass of the rigid body and x_{13} is the height of its centre of gravity in the equilibrium position, we obtain

$$\delta^2 W = \rho g \iint (x_1\gamma_1 + x_2\gamma_2)u_n dS - \frac{1}{2}\left[M_1 g x_{13} + \rho g \int x_3^\circ d\tau\right](\gamma_1^2 + \gamma_2^2) + \frac{1}{2}\rho g \iint u_n^2 dS$$

We now evaluate u_n and, as before, retrieve Rumyantsev's result.

REFERENCE

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Translated by D.L.